AoM 23-24 – RESIT SOLUTIONS

1 S is a smooth manifold of dim $4-2=2$ if $(0,1)$ is a regular value of Φ. In fact in this case it is a 2D embedded submanifold of $\mathbb{R}^4$.

To be a regular value we need to check that $D\Phi$ has maximal rank at all pts such that

$\Phi(p) = (0,1)$

$$\Updownarrow$$

$$\begin{cases} x^2+y=0 \\ x^2+y+y^2+z^2+w^2=1 \end{cases} \Longleftrightarrow \begin{cases} y=-x^2 \\ y^2+z^2+w^2=1 \end{cases}$$

$$D\Phi\Big|_S = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2z & 2w \end{pmatrix}\Bigg|_S = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 1-2x^2 & 2z & 2w \end{pmatrix}\Bigg|_S$$

If $x=y=0 \Rightarrow z^2+w^2=1$ and thus either z or w or both are nonvanishing and

$\begin{pmatrix} 1 & 0 \\ 1 & 2z \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 1 & 2w \end{pmatrix}$ are minors with nonvanishing determinants (so rank 2)

If $x \neq 0 \Rightarrow \det\begin{pmatrix} 2x & 1 \\ 2x & 1-2x^2 \end{pmatrix} = -4x^3+2x-2x$
$= -4x^3 \neq 0$

then rank $=2$

Since $D\Phi|_S$ has max rank 2 on S, $(0,1)$ is a regular value.

2. 1. $\tilde{x}, \tilde{y}$ are coordinates defined by the chart $\varphi(x,y) = (x, x^3+y)$ on $\mathbb{R}^2$. This is invertible by $\varphi^{-1}(\tilde{x},\tilde{y}) = (\tilde{x}, \tilde{y} - \tilde{x}^3)$ and $id_{\mathbb{R}^2} \circ \varphi^{-1}$ is smooth thus is a smooth chart.

2. $\frac{\partial}{\partial x}$ are vector fields defined locally by $\forall f \in C^\infty(M)$

$$\frac{\partial}{\partial x}\Big|_p f = D(id^{-1}_{\mathbb{R}^2} \circ f)|_p e_1$$

$$\frac{\partial}{\partial \tilde{x}}\Big|_p f = D(\varphi^{-1} \circ f)|_p e_1$$

Since they pointwise act as derivations they define tangent vectors. By construction they map $p \mapsto v_p \in T_pM$ & thus are sections of TM. Smoothness follows immediately by the construction.

3. One can use the change of variables rule

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

to find

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial x}{\partial \tilde{x}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \tilde{x}} \frac{\partial}{\partial y} = \frac{\partial \tilde{x}}{\partial \tilde{x}} \frac{\partial}{\partial x} + \frac{\partial(\tilde{y} - \tilde{x}^3)}{\partial \tilde{x}} \frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial x} - 3\tilde{x} \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \tilde{y}} = \frac{\partial x}{\partial \tilde{y}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \tilde{y}} \frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

4. While $\frac{\partial}{\partial \tilde{y}}\Big|_p = \frac{\partial}{\partial y}\Big|_p$ at all points, $\frac{\partial}{\partial \tilde{x}}\Big|_{(1,0)} = \left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\Big|_{(1,0)} \neq \frac{\partial}{\partial x}\Big|_{(1,0)}$

3\. 1. It is known from linear algebra that ker of a linear map is linear subspace, which is what the claim is proving. Otherwise they can explicitly follow the claim

2\. Enough to show linear independence: $aX + bY = 0$ iff $a\partial x = 0$ and $b\partial y = 0 \Rightarrow a = b = 0$

3\. $[X, Y] = Z$

4\. Since $\eta(Z) \equiv 1$, $Z_q \notin D_q \ \forall q \in M$

4\. 1. If Ω is the region bounded by $\pi \circ \gamma$, then $\pi \circ \gamma = \partial \Omega$

This is a line integral so we can use the fundamental thm of calculus from the homework:

$$\int_\gamma \frac{1}{2}(x\,dy - y\,dx) = \int_I \frac{1}{2}(\gamma^1 \dot\gamma^2 - \gamma^2 \dot\gamma^1)\,dt$$

$$= \int_{\pi \circ \gamma} \frac{1}{2}(x\,dy - y\,dx) = \int_\Omega dx \wedge dy$$

integral independent on γ^3

Stokes

2\. Up to rescaling $I = [0,1]$. Condition (a) reads

$$\dot\gamma^3 - \frac{1}{2}(\gamma^1 \dot\gamma^2 - \gamma^2 \dot\gamma^1) = 0 \quad \text{in } [0,t]$$

which can be integrated to yield (as the prev. pt)

$$\gamma^3(t) - \gamma^3(0) = \int_0^t \frac{1}{2}(\gamma^1 \dot\gamma^2 - \gamma^2 \dot\gamma^1) = \int_{\gamma|_{[0,t]}} \frac{1}{2}(x\,dy - y\,dx)$$

3. follows from 2 by using 1 and setting $t=1$

$$0 = \gamma^3(1) - \gamma^3(0) = \int_\gamma \frac{1}{2}(x\,dy - y\,dx) = \int_\Omega dx \wedge dy$$